Int. J. Solids Structures, 1977, Vol. 13, pp. 93–104. Pergamon Press. Printed in Great Britain

RADIAL DEFLECTIONS OF THIN PRECOMPRESSED CYLINDRICAL RUBBER BUSH MOUNTINGS

JAMES M. HILL
Department of Mathematics, The University of Wollongong, Wollongong, New South Wales, Australia

(Received 27 February 1976)

Abstract—For an isotropic incompressible hyperelastic Varga material the plane stress (membrane) theory of thin sheets is employed to formulate the load-deflection relation for small superimposed radial deflections of a cylindrical rubber bush which is precompressed by a large uniform radial inflation. The Varga material is a prototype for rubber over a limited range of deformation and the load-deflection relation obtained provides an extreme lower bound to the practical situation. Moreover this relation complements existing results for cylindrical rubber bushes so that now at least some assessment can be made of the effect of precompression on the radial mode of deflection for bushes of finite length. Typical numerical values are given and are contrasted with corresponding values obtained from existing plane strain radial load-deflection relations for long precompressed cylindrical rubber bush mountings.

1. INTRODUCTION

Rubber bush mountings, consisting of cylindrical rubber tubes bonded on their outer and inner curved surfaces to effectively rigid metal cylinders, are widely used as engineering components. The radial mode of deflection for such bushes is that produced by fixing the outer cylinder and moving the inner cylinder uniformly along its length in a radial direction. For initially unstressed bushes of finite length the radial load-deflection relation is given in [1]. In situations where large radial loads are expected, the rubber is precompressed on assembly in its housing. Plane strain radial load-deflection relations for precompressed bushes which are sufficiently long so that end-effects can be ignored are given in [2, 3] for the neo-Hookean and general isotropic incompressible hyperelastic material respectively. In this paper for a particular elastic material which we refer to as the Varga material we obtain the plane stress radial load-deflection relation for precompressed bushes. This relation gives an extreme lower bound to the practical situation and together with [1–3] provides some theoretical basis from which an assessment could be made of the likely behaviour of precompressed bushes of finite length.

The results obtained in this paper apply to an isotropic incompressible hyperelastic material which has strain-energy function Σ given by

$$\Sigma = 2\mu(\lambda_1 + \lambda_2 + \lambda_3 - 3), \tag{1.1}$$

where μ is the usual infinitesimal shear modulus and λ_i $(i = 1, 2, 3)$ are the principal stretches. This strain-energy function was originally proposed by Varga[4] (page 102) for natural rubber vulcanisates and later by Dickie and Smith[5] for styrene–butadiene vulcanisates. They conclude that such a strain-energy function is a valid prototype for these materials provided the maximum principal stretch does not exceed 2. Indeed, within this range of deformation, there are situations (for example, in simple extension) for which the Varga material gives a closer approximation to experimental results than does the neo-Hookean material. Over this limited range of deformation we would therefore expect the results obtained for the Varga material to be physically meaningful and moreover to be in reasonable agreement with the corresponding results for the neo-Hookean material.

Rubber bushes are effectively precompressed by forcing the rubber tube over the inner cylinder and then forcing the outer cylinder over the rubber. The rubber is either "cold" bonded to the metal cylinders or, for situations where axial loads are not expected to be large, is not bonded at all. For no bonding we assume that the initial compression is sufficiently large so as to prevent slipping between the rubber and the metal cylinders. If we assume that this initial compression can be approximated by a uniform radial inflation then in general for the plane stress (membrane) theory of isotropic incompressible hyperelastic materials we are required to solve a second order nonlinear ordinary differential equation. For the neo-Hookean material this equation has been solved numerically (see for example [6], page 143; [7] and [8]. The main reason for considering the strain-energy function (1.1) is that for this material the second order nonlinear differential equation can be integrated and moreover closed form solutions can be obtained for the small radial deflection which is superimposed upon the initial compression.

In the following section we give the basic equations for the plane stress theory of isotropic incompressible hyperelastic materials. The second order nonlinear ordinary differential equation for the uniform radial inflation is given in Section 3 for the general isotropic incompressible hyperelastic material. For the neo-Hookean material we note that this equation can be reduced to a first order Abel equation (Murphy [9], page 23). This reduction has not been given previously and since the Abel equation obtained is not one of the standard types which can be integrated it would appear that the uniform radial inflation for the neo-Hookean material cannot be expressed in terms of elementary functions. For the Varga material we show that the solution is readily obtained.

In Section 4 we give the governing system of ordinary differential equations for the small radial deformation which is superimposed upon the initial compression. For the general isotropic incompressible hyperelastic material this fourth order system of non-homogeneous linear equations is shown to admit two simple solutions and a first integral. These solutions are consequences of the invariance of the governing equations under translations while the first integral is derived by considering the resultant force acting on a cylinder which was given originally by a right circular cylinder. Evidently in principle these three results are sufficient to reduce the fourth order system to one of first order. However for the general material the governing fourth order system is not readily uncoupled. For the Varga material we show in Section 5 that the system can be solved completely and we give closed form solutions. In Section 6 we use these solutions to formulate the radial load-deflection relation for this material. In the limit of no precompression the result obtained reduces to the linear plane stress radial load-deflection relation. Typical numerical values are given in Section 7.

2. BASIC EQUATIONS FOR THE PLANE STRESS THEORY OF THIN SHEETS OF ISOTROPIC INCOMPRESSIBLE HYPERELASTIC MATERIALS

In this section we give the basic equations for the membrane theory of thin plane sheets of uniform original thickness $2h_0$. The final equilibrium equations in the form of (2.12) or (2.17) have not been given previously and so we give a brief derivation. This is most easily done in terms of general plane curvilinear coordinates.

For material coordinates X^K $(K = 1, 2, 3)$ and spatial coordinates x^i $(i = 1, 2, 3)$ we consider, as is usual in plane stress theory, the deformation

$$x^a = x^a(X^A), \qquad x^3 = \lambda(X^A)X^3, \tag{2.1}$$

where X^A $(A = 1, 2)$ and x^a $(a = 1, 2)$ are general plane curvilinear coordinates and X^3 and x^3 are rectangular cartesian coordinates. If G_{AB} and g_{ab} are the metric tensors of the coordinates X^A and x^a respectively and if

$$G = |G_{AB}|, \qquad g = |g_{ab}|, \tag{2.2}$$

then the deformation (2.1) of an incompressible material satisfies the condition

$$\frac{\partial(x^1, x^2)}{\partial(X^1, X^2)} = \sqrt{\left(\frac{G}{g}\right)}\Lambda, \tag{2.3}$$

where

$$\frac{\partial(x^1, x^2)}{\partial(X^1, X^2)} = \frac{\partial x^1}{\partial X^1}\frac{\partial x^2}{\partial X^2} - \frac{\partial x^2}{\partial X^1}\frac{\partial x^1}{\partial X^2}, \tag{2.4}$$

and we have introduced $\Lambda(X^A)$ which is defined by

$$\Lambda = \lambda^{-1}. \tag{2.5}$$

If we make the usual assumptions of membrane theory (see Green and Adkins[6], page 126) we can show for an isotropic incompressible hyperelastic material that the Cauchy stress resultants T^{ab} for the deformation (2.1) are given by

$$T^{ab} = 2h_0\lambda(\Lambda\psi g^{ab} + \phi c^{-1ab}), \tag{2.6}$$

where c^{-1ab} is the inverse Cauchy deformation tensor which is defined by

$$c^{-1ab} = G^{AB}x^a_{;A}x^b_{;B}, \tag{2.7}$$

where semi-colons are used here to denote the total covariant derivative. Also ϕ and ψ are given by

$$\phi = 2\frac{\partial \Sigma}{\partial I}, \qquad \psi = \frac{\partial \Sigma}{\partial \Lambda}, \tag{2.8}$$

where $\Sigma(I, \Lambda)$ is the strain-energy function and $I = c_a^{-1a}$. If we use the Euler–C. Neumann identity

$$(\Lambda X^A_{;a})_{;A} = 0, \tag{2.9}$$

then by means of the first Piola–Kirchoff stress tensor $T_R{}^{Ab}$ which is given by

$$T_R^{Ab} = \Lambda X^A_{;a}T^{ab}, \tag{2.10}$$

we can show from (2.6) that the equilibrium equations

$$T^{ab}_{;a} = 0, \qquad T^{Ab}_{R;A} = 0, \tag{2.11}$$

become

$$\Lambda\psi_{;a}g^{ab} + \phi(\nabla^2 x^b + \Gamma^b_{ad}c^{-1ad}) + G^{AB}\phi_{;A}x^b_{;B} = 0, \tag{2.12}$$

where ∇^2 is given by

$$\nabla^2 = G^{AB}\left[\frac{\partial^2}{\partial X^A \partial X^B} - \Gamma^D_{AB}\frac{\partial}{\partial X^D}\right], \tag{2.13}$$

and Γ^D_{AB} and Γ^b_{ad} are the Christoffel symbols based on the metric tensors G_{AB} and g_{ab} respectively. We note here that the equilibrium equations (2.12) in rectangular cartesian coordinates are given by Wong and Shield[7] for the particular case of the neo-Hookean material.

Throughout the remainder of the paper we use material and spatial cylindrical polar coordinates (R, Θ, Z) and (r, θ, z) respectively. The deformation (2.1) and the condition of incompressibility (2.3) become

$$r = r(R, \Theta), \qquad \theta = \theta(R, \Theta), \qquad z = \lambda(R, \Theta)Z, \tag{2.14}$$

$$\frac{\partial(r, \theta)}{\partial(R, \Theta)} = \frac{R}{r}\Lambda, \tag{2.15}$$

while the invariant I can be shown to be given by

$$I = r_R{}^2 + \frac{r_\Theta{}^2}{R^2} + r^2\left(\theta_R{}^2 + \frac{\theta_\Theta{}^2}{R^2}\right), \tag{2.16}$$

where subscripts denote partial differentiation. From (2.12) we obtain

$$\phi\left\{\nabla^2 r - r\left(\theta_R{}^2 + \frac{\theta_\Theta{}^2}{R^2}\right)\right\} + \phi_R r_R + \frac{\phi_\Theta r_\Theta}{R^2} + \frac{r}{R}\frac{\partial(\psi, \theta)}{\partial(R, \Theta)} = 0,$$
$$\phi\left\{\nabla^2\theta + \frac{2}{r}\left(r_R\theta_R + \frac{r_\Theta\theta_\Theta}{R^2}\right)\right\} + \phi_R\theta_R + \frac{\phi_\Theta\theta_\Theta}{R^2} - \frac{1}{rR}\frac{\partial(\psi, r)}{\partial(R, \Theta)} = 0, \tag{2.17}$$

where ∇^2 is the usual Laplacian in cylindrical polar coordinates, namely

$$\nabla^2 = \frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial\Theta^2}. \tag{2.18}$$

We note also that using the first Piola–Kirchoff stress tensor given by (2.10) we can show that the resultant force in the direction $\theta = 0$, which must be applied to an originally right circular cylinder R = constant, is given by

$$F = -2h_0\int_0^{2\pi} [\psi(r\sin\theta)_\Theta + R\phi(r\cos\theta)_R]\,d\Theta. \tag{2.19}$$

3. THE INITIAL DEFORMATION

We suppose that the initial compression of the bush which is originally of inner and outer radii A and B respectively, is effected by the uniform radial inflation

$$r = f(R), \qquad \theta = \Theta, \qquad z = \lambda(R)Z, \tag{3.1}$$

where f and λ are functions of R only. From (2.15)–(2.17) and (3.1) we obtain

$$\Lambda = \frac{ff'}{R}, \qquad I = f'^2 + \frac{f^2}{R^2},$$
$$\phi\left(f'' + \frac{f'}{R} - \frac{f}{R^2}\right) + \phi' f' + \psi'\frac{f}{R} = 0, \tag{3.2}$$

where primes denote differentiation with respect of R. In general these equations constitute a highly nonlinear second order differential equation for the function $f(R)$. As mentioned in the introduction this equation for the neo-Hookean material has been solved numerically by a number of authors. Before considering the Varga material (1.1) it would seem worthwhile noting that in the case of the neo-Hookean material the nonlinear equation can be reduced to a first order Abel equation.

For the neo-Hookean material the response coefficients ϕ and ψ are given by

$$\phi = \mu, \qquad \psi = -\mu\Lambda^{-3}, \tag{3.3}$$

where the constant μ is the linear shear modulus. From $(3.2)_3$ and (3.3) we obtain

$$\frac{d}{dR}\left(f' + \frac{f}{R}\right) = \frac{f}{R}\frac{d}{dR}(\Lambda^{-3}). \tag{3.4}$$

If now we make the transformation

$$\frac{f}{R} = [x(1-xy)]^{-1/2}, \qquad \Lambda = x^{-1}, \tag{3.5}$$

so that from $(3.2)_1$ we obtain

$$f' = \left[\frac{(1-xy)}{x}\right]^{1/2}, \tag{3.6}$$

then on substituting (3.5) and (3.6) into (3.4) we obtain

$$y\frac{dy}{dx} = \left(\frac{2}{x^3}+6\right) - \left(\frac{3}{x^2}+6x\right)y. \tag{3.7}$$

This is an Abel equation of the second kind (Murphy [9], page 25) with special solution x^{-1}. It can be transformed into an Abel equation of the first kind if we take $y_1 = y^{-1}$ as the dependent variable. Neither of these Abel equations are of the standard types which can be integrated and thus an approximate method of solution is probably required. For the problem under consideration it is desirable that some analytic expression is obtained for the deformation (3.1) and so we shall not consider further approximate solutions of (3.7).

For the Varga material the strain-energy function (1.1) in terms of I and Λ is given by

$$\Sigma = 2\mu[(I+2\Lambda)^{1/2} + \Lambda^{-1} - 3], \tag{3.8}$$

so that from (2.8) the response coefficients are given by

$$\phi = 2\mu(I+2\Lambda)^{-1/2}, \qquad \psi = 2\mu(I+2\Lambda)^{-1/2} - 2\mu\Lambda^{-2}, \tag{3.9}$$

and thus we have the relation

$$\psi = \phi - 2\mu\Lambda^{-2}. \tag{3.10}$$

From $(3.2)_3$ and (3.10) we obtain

$$\frac{d}{dR}\left\{\phi\left(f' + \frac{f}{R}\right)\right\} = 2\mu\frac{f}{R}\frac{d}{dR}(\Lambda^{-2}). \tag{3.11}$$

But from $(3.2)_1$, $(3.2)_2$ and $(3.9)_1$ we have ϕ is given by

$$\phi = 2\mu\left(f' + \frac{f}{R}\right)^{-1}, \tag{3.12}$$

and thus from (3.11) and (3.12) we see that Λ can at most be a constant. Thus from $(3.2)_1$ we see that the deformation (3.1) for the Varga material becomes

$$r = (\alpha R^2 + \beta)^{1/2}, \qquad \theta = \Theta, \qquad z = \alpha^{-1}Z, \tag{3.13}$$

where α and β are constants.

Thus for the Varga material the uniform radial inflation (3.1) is the same as the plane strain uniform radial inflation given by Rivlin [10]. This result is not as unreasonable as it would appear at first sight. For the neo-Hookean material Wong and Shield [7] have shown that for moderate deformations of a specific boundary value problem the exact numerical solution of $(3.2)_1$ and (3.4) can be accurately approximated by the function

$$f(R) = c_1R + c_2R^{-1}, \tag{3.14}$$

where c_1 and c_2 are constants determined by the boundary conditions of $f(R)$. We see from $(3.13)_1$ that if $K = \beta/\alpha$ is small so that we can neglect terms of order K^2 then up to order K we obtain a function of the form (3.14). Bearing in mind the close behaviour of the neo-Hookean and Varga theories for deformations with maximum principal stretch less than 2 it is therefore not unreasonable that (3.13) describes the uniform radial inflation provided of course we restrict our attention to this range of deformation for which we know the Varga material to be physically meaningful. If we take (3.13) to describe the initial compression then there is the additional advantage that the plane stress radial load-deflection relation can be easily contrasted with the plane strain relations given in [2] and [3].

Finally in this section we remark that considering the numerical results of Wong and Shield [7] it is tempting to use (3.14) for the neo-Hookean material in the analysis of the following sections. However, if we were to do this consistently the final load-deflection relation would only be valid for deformations which are no larger than those for which the Vagra material is physically meaningful.

4. GOVERNING SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS FOR THE SMALL SUPERIMPOSED DEFORMATION

In this section for an arbitrary isotropic incompressible hyperelastic material we give the fourth order system of ordinary differential equations for the small radial deflection which is superimposed upon the initial compression (3.1). Although we shall only solve this system completely for the Varga material it would seen worthwhile noting that for the general material two simple solutions and an independent first integral can be given.

We suppose that superimposed upon the deformation (3.1) the inner metal cylinder of the bush is displaced a small distance ϵ uniformly along its length in the direction of $\Theta = 0$ while the outer cylinder is held fixed. The displacement boundary conditions at the inner cylinder suggest we look for solutions of (2.15) and (2.17) of the form

$$\begin{aligned} r &= f(R) + \epsilon u(R)\cos\Theta, \\ \theta &= \Theta + \epsilon v(R)\sin\Theta, \\ \Lambda &= \Lambda_0(R) + \epsilon\Lambda_1(R)\cos\Theta, \end{aligned} \tag{4.1}$$

where u, v and Λ_1 are functions of R only and $f(R)$ and $\Lambda_0(R)$ are assumed to be solutions of (3.2). Moreover in the following we use the subscript zero to denote quantities evaluated at the initial deformation. The displacement boundary conditions at the inner and outer cylinders now become

$$\begin{aligned} u(A) &= 1, \qquad v(A) = -\frac{1}{f(A)} \\ u(B) &= 0, \qquad v(B) = 0. \end{aligned} \tag{4.2}$$

We remark here that, as is usual in plane stress theory, no displacement boundary conditions are imposed upon Λ.

From (2.16) and (4.1) we find on neglecting terms of order ϵ^2 that

$$I = I_0 + \epsilon I_1 \cos\Theta, \tag{4.3}$$

where I_1 is given by

$$I_1 = 2\left(f'u' + \frac{fu}{R^2} + \frac{f^2 v}{R^2}\right). \tag{4.4}$$

Also from (2.8), $(4.1)_3$ and (4.3) we obtain

$$\begin{aligned} \phi &= \phi_0 + \epsilon\phi_1 \cos\Theta, \\ \psi &= \psi_0 + \epsilon\psi_1 \cos\Theta, \end{aligned} \tag{4.5}$$

where ϕ_1 and ψ_1 are given by

$$\begin{aligned} \phi_1 &= I_1 \frac{\partial\phi_0}{\partial I_0} + \Lambda_1 \frac{\partial\phi_0}{\partial\Lambda_0}, \\ \psi_1 &= I_1 \frac{\partial\psi_0}{\partial I_0} + \Lambda_1 \frac{\partial\psi_0}{\partial\Lambda_0}. \end{aligned} \tag{4.6}$$

We remind the reader that Λ_0 and I_0 are assumed to be given by $(3.2)_1$ and $(3.2)_2$ respectively while ϕ_0 and ψ_0 are the response coefficients evaluated at the initial deformation (3.1) and defined by (2.8).

If we use (4.1) and (4.5) in the condition of incompressibility (2.15) and the equations of eqilibrium (2.17) then we obtain on equating terms of order ϵ,

$$\begin{gathered} fu' + f'u + ff'v = \Lambda_1 R, \\ \phi_0\left(u'' + \frac{u'}{R} - \frac{2u}{R^2} - \frac{2fv}{R^2}\right) + \phi_1\left(f'' + \frac{f'}{R} - \frac{f}{R^2}\right) + \phi_0' u' + \phi_1' f' + \frac{\psi_0'}{R}(u + fv) + \frac{\psi_1' f}{R} = 0, \\ \phi_0\left(v'' + \frac{v'}{R} - \frac{v}{R^2} + \frac{2f'}{f}v' - \frac{2u}{fR^2}\right) - \frac{\phi_1}{R^2} + \phi_0' v' + \frac{\psi_0' u}{fR} - \frac{\psi_1 f'}{fR} = 0. \end{gathered} \tag{4.7}$$

Moreover from (2.19) we find that up to first order in ϵ the force F is given by

$$F = -2h_0\epsilon\pi[\psi_1 f + R\phi_0(u - fv)' + R\phi_1 f']. \tag{4.8}$$

Assuming that f and Λ_0 are known functions the eqns (4.7) constitute three equations for the determination of u, v and Λ_1. In the following section we solve this system completely for the special case of the Varga material (1.1).

For the general material we note the following linearly independent solutions,

$$\begin{aligned} u &= 1, \quad & v &= -\frac{1}{f}, \quad & \Lambda_1 &= 0, \\ u &= f', \quad & v &= -\frac{1}{R}, \quad & \Lambda_1 &= \Lambda_0'. \end{aligned} \tag{4.9}$$

If we denote material and spatial rectangular cartesian coordinates by (X, Y, Z) and (x, y, z) respectively then the reader can easily verify that the above two solutions arise from the invariance of the condition of incompressibility and the equations of equilibrium under the respective translations

$$\begin{gathered} (x, y, z) \to (x + \epsilon, y, z), \\ (X, Y, Z) \to (X + \epsilon, Y, Z). \end{gathered} \tag{4.10}$$

In addition to the solutions (4.9) we have the independent first integral

$$\psi_1 f + R\phi_0(u - fv)' + R\phi_1 f' = \text{constant}, \tag{4.11}$$

which can be deduced directly from (4.8) or from $(4.7)_2$ and $(4.7)_3$. The solutions (4.9) and the first integral (4.11) are in principle sufficient to reduce the fourth order system (4.7) to one of first order. However, for the general material there appears to be no simple way of uncoupling (4.7). In the remainder of the paper we shall only consider (4.7) for the Varga material for which a further integration can be effected.

5. SOLUTIONS FOR THE VARGA MATERIAL

For the Varga material the initial deformation is given by (3.13) so that f and Λ_0 in the equations of the previous section are given by

$$f(R) = (\alpha R^2 + \beta)^{1/2}, \qquad \Lambda_0(R) = \alpha, \tag{5.1}$$

where α and β are constants which we assume to be determined by the initial and final radii of the tube before and after precompression. From (3.10), $(4.1)_3$, (4.5) and $(5.1)_2$ we obtain the relation

$$\psi_1 = \phi_1 + 4\mu\alpha^{-3}\Lambda_1. \tag{5.2}$$

If we use (4.4) and the condition of incompressibility $(4.7)_1$ then from $(4.6)_1$ we can deduce

$$\phi_1 = -\phi_0\left(f' + \frac{f}{R}\right)^{-1}\left(u' + \frac{u}{R} + \frac{f}{R}v\right), \tag{5.3}$$

where we have used $(3.9)_1$ and the expression (3.12) for ϕ_0. If now in (4.11) we set the constant equal to $\gamma_1 K$ where γ_1 is an arbitrary constant and $K = \beta/\alpha$ then from (4.11), (5.2) and (5.3) we obtain

$$4\mu\alpha^{-3}f\Lambda_1 = \phi_0[(Rfv)' + u] + \gamma_1 K. \tag{5.4}$$

From the above equations and noting from (3.10) that $\psi_0' = \phi_0'$ we can show that $(4.7)_2$ can be integrated to give

$$\phi_0[(Rfv)' + u] = \gamma_1 R^2 + \gamma_2 R(R^2 + K)^{1/2}, \tag{5.5}$$

where γ_2 is a further arbitrary constant. From this equation we see that (5.4) becomes

$$\Lambda_1(R) = (4\mu)^{-1}\alpha^{5/2}[\gamma_1(R^2 + K)^{1/2} + \gamma_2 R]. \tag{5.6}$$

Thus $\Lambda_1(R)$ is known and from $(4.7)_1$, $(5.1)_1$ and (5.6) we have

$$Rfv = (4\mu)^{-1}\alpha^2 R(R^2 + K)^{1/2}[\gamma_1(R^2 + K)^{1/2} + \gamma_2 R] - Ru - (R^2 + K)u'. \tag{5.7}$$

Substitution of (5.7) into (5.5) yields a first order differential equation in u' which can be readily integrated.

Omitting the details the final result is

$$\begin{aligned}2\mu u(R) = {} & \frac{\gamma_1}{16}\alpha^2\left\{3R^2 + \frac{KR}{(R^2 + K)^{1/2}}\log[R + (R^2 + K)^{1/2}]\right\} \\ & + \frac{\gamma_2}{16}\alpha^2\{3R(R^2 + K)^{1/2} - K\log[R + (R^2 + K)^{1/2}]\} \\ & - \frac{\gamma_1}{4}\alpha^{1/2}\{R(R^2 + K)^{1/2} - K\log[R + (R^2 + K)^{1/2}]\} \\ & - \frac{\gamma_2}{4}\alpha^{1/2}\left\{R^2 + \frac{KR}{(R^2 + K)^{1/2}}\log[R + (R^2 + K)^{1/2}]\right\} \\ & + \frac{\gamma_3 R}{(R^2 + K)^{1/2}} + \gamma_4,\end{aligned} \tag{5.8}$$

where γ_3 and γ_4 are integration constants. From (5.7) and (5.8) we find that the solution for v is given by

$$\begin{aligned}2\mu\alpha^{1/2}v(R) = {} & -\frac{\gamma_1}{16}\alpha^2\left\{\frac{(R^2 - K)}{(R^2 + K)^{1/2}} + \frac{K}{R}\log[R + (R^2 + K)^{1/2}]\right\} \\ & - \frac{\gamma_2}{16}\alpha^2\left\{\frac{(R^2 + 2K)}{R} - \frac{K}{(R^2 + K)^{1/2}}\log[R + (R^2 + K)^{1/2}]\right\} \\ & + \frac{\gamma_1}{4}\alpha^{1/2}\left\{3R - \frac{K}{(R^2 + K)^{1/2}}\log[R + (R^2 + K)^{1/2}]\right\} \\ & + \frac{\gamma_2}{4}\alpha^{1/2}\left\{3(R^2 + K)^{1/2} + \frac{K}{R}\log[R + (R^2 + K)^{1/2}]\right\} \\ & - \frac{\gamma_3}{R} - \frac{\gamma_4}{(R^2 + K)^{1/2}}.\end{aligned} \tag{5.9}$$

We note that the solutions associated with the constants γ_3 and γ_4 are those given by $(4.9)_2$ and $(4.9)_1$ respectively. In the following section we use the above solutions to formulate the radial load-deflection relation.

6. RADIAL LOAD-DEFLECTION RELATION FOR THE VARGA MATERIAL

In terms of the constant γ_1 we have from (4.8) that the force F required to maintain the radial deflection is given by

$$F = -2\pi\epsilon h_0 K\gamma_1. \tag{6.1}$$

From the boundary conditions (4.2) and the solutions (5.8) and (5.9) we obtain, after a long but straightforward calculation to determine γ_1, the following relation,

$$F = \frac{128\pi\mu\epsilon K^2 h_0[\omega_1\alpha^2 + \omega_2\alpha^{1/2}]}{[\Omega_1\alpha^4 + \Omega_2\alpha^{5/2} + \Omega_3\alpha]}, \tag{6.2}$$

where ω_1, ω_2, Ω_1, Ω_2 and Ω_3 are defined by

$$\begin{aligned}
\omega_1 &= [(B^2+K)^2 - (A^2+K)^2],\\
\omega_2 &= 2[\sigma(B) - \sigma(A)] + 8[B(B^2+K)^{3/2} - A(A^2+K)^{3/2}],\\
\Omega_1 &= 4(B^4 - A^4)[(B^2+K)^2 - (A^2+K)^2] - [\sigma(B) - \sigma(A)]^2,\\
\Omega_2 &= 32[B^2(B^2+K) - A^2(A^2+K)][\sigma(B) - \sigma(A) + B(B^2+K)^{1/2}(2B^2+K)\\
&\quad - A(A^2+K)^{1/2}(2A^2+K)] + 32K^2(B^2-A^2)[B(B^2+K)^{1/2} - A(A^2+K)^{1/2}],\\
\Omega_3 &= 16[\sigma(B) - \sigma(A) + 4B^3(B^2+K)^{1/2} - 4A^3(A^2+K)^{1/2}][\sigma(B) - \sigma(A)\\
&\quad + 4B(B^2+K)^{3/2} - 4A(A^2+K)^{3/2}] - 64[B^2(B^2+K) - A^2(A^2+K)]^2,
\end{aligned} \tag{6.3}$$

where the function $\sigma(R)$ is defined by

$$\sigma(R) = K^2\log[R + (R^2+K)^{1/2}] - R(R^2+K)^{1/2}(2R^2+K). \tag{6.4}$$

For the Varga material (6.2), (6.3) and (6.4) constitute the "exact" plane stress radial load-deflection relation for thin precompressed rubber bushes. If we set $\alpha = 1$ and let K tend to zero then we obtain from the above equations the well known linear plane stress relation (see for example [1])

$$F_0 = \frac{160\pi\mu\epsilon h_0(A^2+B^2)}{[25(A^2+B^2)\log(B/A) - 9(B^2-A^2)]}, \tag{6.5}$$

where F_0 is the force required to maintain the deformation for no precompression.

In the following section we illustrate (6.2) with typical numerical values of F/F_0 which we compare with the corresponding ratios for long precompressed cylindrical bushes. For the case of long bushes the initial compression is given by (3.13) (see [2] and [3]). From the general formula given in [3] for the arbitrary isotropic incompressible hyperelastic material we can show that for the Varga material we have

$$F_V^* = 64\pi\mu\epsilon K^2\alpha^{-1/2}L\frac{\omega_2}{\Omega_3}, \tag{6.6}$$

while for the neo-Hookean material we have

$$F_N^* = 16\pi\mu\epsilon K^2 L\frac{\omega_1}{\Omega_1}, \tag{6.7}$$

where L is the original length of the long rubber tube before precompression and ω_1, ω_2, Ω_1 and

Ω_3 are as defined above. We remark that (6.7) is given explicitly in [2]. For completeness we note that the standard linear plane strain formula for radial deflections of long bushes, that is (see [1])

$$F_0^* = \frac{4\pi\mu\epsilon L(A^2+B^2)}{[(A^2+B^2)\log(B/A)-(B^2-A^2)]}, \tag{6.8}$$

and we note that this result can be obtained from both of (6.6) and (6.7) in the limit as K tends to zero and $\alpha = 1$.

7. NUMERICAL RESULTS

The principal stretches of the initial deformation (3.13) are given by

$$\lambda_1 = \frac{\alpha^{1/2}R}{(R^2+K)^{1/2}}, \qquad \lambda_2 = \frac{\alpha^{1/2}(R^2+K)^{1/2}}{R}, \qquad \lambda_3 = \frac{1}{\alpha}, \tag{7.1}$$

where $K = \beta/\alpha$. For α and K positive the maximum values of these stretches are given by

$$\lambda_{1\,\max} = \frac{\alpha^{1/2}B}{(B^2+K)^{1/2}}, \qquad \lambda_{2\,\max} = \frac{\alpha^{1/2}(A^2+K)^{1/2}}{A}, \qquad \lambda_{3\,\max} = \frac{1}{\alpha}. \tag{7.2}$$

In order to illustrate the results of the previous section we suppose that the precompression is effected by increasing the internal radius A to a and leaving the external radius B unaltered. If we define δ and γ by

$$\delta = \frac{B}{A}, \qquad \gamma = \frac{a}{A}, \tag{7.3}$$

then since $a \geq A$ and $B > a$ we have

$$1 \leq \gamma < \delta. \tag{7.4}$$

Moreover the constants α and K are given by

$$\alpha = \frac{(\delta^2-\gamma^2)}{(\delta^2-1)}, \qquad K = \frac{B^2(\gamma^2-1)}{(\delta^2-\gamma^2)}. \tag{7.5}$$

Typical numerical values of F/F_0, F_V^*/F_0^* and F_N^*/F_0^* are given in Tables 1(a), 1(b) and 1(c) for various values of γ for which the maximum principal stretches as given by (7.2) are less than 2.

The first column of Table 1 gives the value of γ while the second column gives the value of F/F_0 which is obtained from (6.2) and (6.5). The last two columns give the ratios F_V^*/F_0^* and F_N^*/F_0^* which are obtained from (6.6), (6.7) and (6.8). These last two columns are included, firstly to contrast the "compression factor" for the plane stress and plane strain radial load-deflection relations and secondly to give some indication of the differences between the neo-Hookean and Varga theories over this range of deformation. For the data of Table 1(c) we show in Fig. 1 the overall variation of the ratios for the range of deformation having maximum principal stretch less than 2.2.

As far as engineering purposes are concerned these numerical results indicate that the "compression factor" for short bushes is very nearly that for long bushes provided the maximum values of the stretches given by (7.2) do not exceed 2. Moreover in assessing the "compression factor" for bushes of finite length a reasonable approximation would be to take some value which is intermediate to the plane stress and plane strain values.

8. CONCLUSION

For cylindrical rubber bush mountings which are precompressed by a large uniform radial inflation we have derived the plane stress load-deflection relation (6.2) for small superimposed radial deflections for the particular case of the isotropic incompressible Varga material. This

Table 1. Values of F/F_0, F_V^*/F_0^* and F_N^*/F_0^* for various values of a/A

(a) $A = 1.0$, $B = 1.5$

a/A	F/F_0	F_V^*/F_0^*	F_N^*/F_0^*
1.05	1.31	1.33	1.28
1.10	1.77	1.81	1.67
1.15	2.51	2.56	2.26
1.20	3.78	3.76	3.19
1.25	6.20	5.88	4.75

(b) $A = 1.0$, $B = 2.0$

a/A	F/F_0	F_V^*/F_0^*	F_N^*/F_0^*
1.1	1.29	1.34	1.30
1.2	1.72	1.84	1.73
1.3	2.40	2.60	2.38
1.4	3.56	3.86	3.42
1.5	5.74	6.06	5.21

(c) $A = 1.0$, $B = 3.0$

a/A	F/F_0	F_V^*/F_0^*	F_N^*/F_0^*
1.2	1.27	1.34	1.32
1.4	1.66	1.84	1.79
1.6	2.29	2.61	2.51
1.8	3.34	3.88	3.70
2.0	5.30	6.11	5.78

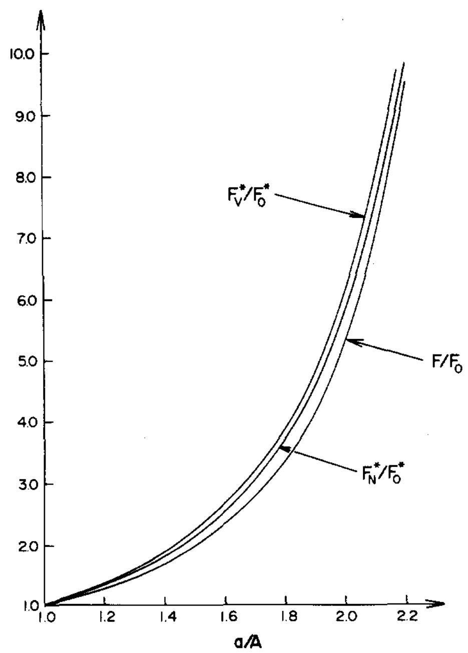

Fig. 1. Variation of the ratios F/F_0, F_V^*/F_0^* and F_N^*/F_0^* for values of a/A for which the maximum principal stretch of (3.13) is less than 2.2 and $A = 1.0$ and $B = 3.0$.

material has strain-energy function given by (1.1) and has been shown by previous authors to be a valid prototype for rubber provided the maximum principal stretch of the deformation does not exceed 2. Within this range of deformation this material is in reasonable agreement with the neo-Hookean material which is the standard prototype for rubber. The relation (6.2) together with results given in [1–3] enables at least some assessment to be made of the likely behaviour of precompressed bushes of finite length. Numerical results indicate that the "compression factors" for plane stress and plane strain radial load-deflection relations are as far as engineering purposes are concerned, very nearly the same.

Acknowledgement—The author wishes to thank Dr. P. B. Lindley (Malaysian Rubber Producers' Research Association, Hertford, England) for suggesting the problem.

REFERENCES

1. J. M. Hill, Radial deflections of rubber bush mountings of finite lengths. *Int. J. Engng Sci.* **13**, 407 (1975).
2. J. M. Hill, The effect of precompression on the load-deflection relations of long rubber bush mountings. *J. Appl. Polymer Sci.* **19**, 747 (1975).
3. J. M. Hill, Closed form solutions for small deformations superimposed upon the simultaneous inflation and extension of a cylindrical tube. *J. Elasticity* **6**(2), 113 (1976).
4. O. H. Varga, *Stress-strain Behaviour of Elastic Materials.* Interscience, New York (1966).
5. R. A. Dickie and T. L. Smith, Viscoelastic properties of a rubber vulcanizate under large deformations in equal biaxial tension, pure shear and simple tension. *Trans. Soc. Rheol.* **15**, 91 (1971).
6. A. E. Green and J. E. Adkins, *Large Elastic Deformations and Non-linear Continuum Mechanics.* Oxford University Press (1960).
7. F. S. Wong and R. T. Shield, Large plane deformations of thin elastic sheets of neo-Hookean material, *Z. Angew. Math. Phys.* **20**, 176 (1969).
8. W. H. Yang, Stress concentration in a rubber sheet under axially symmetric stretching, *J. Appl. Mech.* **34**, 942 (1967).
9. G. M. Murphy, *Ordinary Differential Equations and their Solutions.* Van Nostrand, New York (1960).
10. R. S. Rivlin, Large elastic deformations of isotropic materials, Part VI, Further results in the theory of torsion, shear and flexure. *Phil. Trans. Roy. Soc. Lond.* **A242**, 173 (1949).